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CONTACT PROBLEMS FOR AN ELASTIC SEMI-INFINITE CYLINDER

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A system of homogeneous solutions is constructed for the axisymmetric mixed problem of elasticity theory for an infinite cylinder, one part of whose surface is stress-free, while the other is under sliding boundary conditions. Asymptotic formulas governing the stress concentration and the shape of the free surface at the line of boundary condition separation are obtained. The system can be utilized to satisfy conditions on the endfaces of a semi-infinite or finite cylinder.

Three contact problems of a semi-infinite cylinder partially compressed without friction by an absolutely rigid collar. The conditions on the side surface are hence satisfied exactly. The coefficients in the series of homogeneous solutions are determined from the normal systems of algebraic equations.

1. Let us consider a system of homogeneous solutions each of which satisfies mixed conditions on the surface of cylinder of unit radius

$$\tau_{rz} = u = 0 \quad \text{for} \quad r = 1, \ z \geqslant 0 \tag{1.1}$$

$$\tau_{rz} = \sigma_r = 0 \quad \text{for} \quad r = 1, \ z < 0 \tag{1.2}$$

and has finite elastic stress energy at the line of separation of these conditions at $r = 1$

$$\sigma_r \sim O(z^{\alpha_1 - 1}) \quad \text{for} \quad z \to +0, \qquad u \sim O(z^{\alpha_2}) \quad \text{for} \quad z \to -0 \tag{1.3}$$

$$(\alpha_1, \ \alpha_2 > 0)$$

Let us start with the construction of a subsystem of solutions which increase without limit as $z \to \infty$. In the Papkovich-Neuber formulas [1]

$$u(z,r) = F_1 - \frac{1}{4(1-\sigma)} \frac{\partial}{\partial r}(rF_1 + zF_2 + F_3) \tag{1.4}$$

$$w(z,r) = F_2 - \frac{1}{4(1-\sigma)} \frac{\partial}{\partial z}(rF_1 + zF_2 + F_3)$$

let us set

$$F_1 = 0, \qquad F_2 = A_k e^{a_k z} J_0(a_k r), \qquad F_3 = B_k e^{a_k z} J_0(a_k r)$$
$$(k = 1, 2, \ldots)$$

where A_k and B_k are arbitrary real constants, a_k are positive zeros of the Bessel function $J_1(\nu)$.

The solution obtained, which we denote by the superscript $k1$, satisfy condition (1.1)

$$u^{k1}(z,r) = m_1 a_k e^{a_k z}(A_k z + B_k) J_1(a_k r)$$

$$\tau_{rz}^{k1}(z,r) = 2Gm_1 a_k e^{a_k z}[A_k(a_k z - m_2) + B_k a_k] J_1(a_k r) \tag{1.5}$$

$$\sigma_r^{k1}(z,r) = 2Gm_1 a_k e^{a_k z}\{A_k[2\sigma J_0(a_k r) + [a_k J_0(a_k r) -$$
$$- r^{-1}J_1(a_k r)]z] + B_k[m_2^{-1}(a_k - \sigma a_k - \sigma) J_0(a_k r) - r^{-1}J_1(a_k r)]\}$$

Here

$$m_1 = [4(1-\sigma)]^{-1}, \quad m_2 = 1 - 2\sigma, \quad m_3 = [2(1+\sigma)]^{-1}, \quad m_4 = 2(1-\sigma)$$

In order to satisfy condition (1.2), we find, and add to (1.5), the solution of the following problem:

$$\tau_{rz} = 0 \quad \text{for } r = 1, \ -\infty < z < \infty \tag{1.6}$$

$$u = 0 \quad \text{for} \quad r = 1, \ z \geqslant 0; \qquad \sigma_r = -\sigma_r^{k1}(z, 1) \quad \text{for} \quad r = 1, \ z < 0 \tag{1.7}$$

In (1.4) let us set

$$F_2 = 0, \qquad F_1 = \partial F_4/\partial r, \qquad F_3 = 4(1-\sigma)(F_4 - F_5) \tag{1.8}$$

$$\left(\Delta F_4 = \Delta F_5 = 0, \qquad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)$$

and let us apply the two-sided Laplace transform. Integrating by parts and considering the displacements bounded as $z \to \infty$, decrease as $e^{\alpha z}$ $(\alpha > 0)$ for $z \to -\infty$, and the parameter ν to be within the strip $0 < \operatorname{Re} \nu < \alpha$, we obtain

$$u(\nu, r) = \int_{-\infty}^{\infty} u(z,r) e^{-\nu z} dz = C_k(\nu)[\varepsilon'(\nu) + \rho'(\nu)] \tag{1.9}$$

$$w(\nu, r) = \int_{-\infty}^{\infty} w(z,r) e^{-\nu z} dz = C_k(\nu)\nu[\varepsilon(\nu) - \rho(\nu)] \tag{1.10}$$

$$\varepsilon(\nu) = \nu[J_0(\nu)J_0(\nu r) + rJ_1(\nu)J_1(\nu r)], \quad \rho(\nu) = m_4 J_1(\nu) J_0(\nu r)$$

The prime here denotes the derivative with respect to r, and condition (1.6) has been utilized in calculating the functions $\varepsilon(\nu)$ and $\rho(\nu)$.

Let us evaluate the function $C_k(\nu)$. To do this, let us introduce the notation

$$u^-(\nu) = \int_{-\infty}^{0} u(z,1)\, e^{-\nu z} dz, \quad \sigma^-(\nu) = \int_{-\infty}^{0} \sigma_r(z,1)\, e^{-\nu z} dz$$

$$\sigma^+(\nu) = \int_{0}^{\infty} \sigma_r(z,1)\, e^{-\nu z} dz$$

and let us form the following system of equations from the conditions (1.7):

$$-m_4 \nu J_1^2(\nu) C_k(\nu) = u^-(\nu), \qquad 2G\nu R(\nu)\, C_k(\nu) = \sigma^-(\nu) + \sigma^+(\nu) \quad (1.11)$$

in which according to (1.5)

$$\sigma^-(\nu) = 2Gm_1 a_k (a_k - \nu)^{-1} \{A_k [a_k (a_k - \nu)^{-1} J_0(a_k) - 2\sigma J_0(a_k)] + B_k m_2^{-1} (\sigma a_k + \sigma - a_k) J_0(a_k)\} \quad (1.12)$$

$$R(\nu) = \nu^2 J_0^2(\nu) + (\nu^2 - m_4) J_1^2(\nu) \quad (1.13)$$

Eliminating the function $C_k(\nu)$ from (1.11), we obtain a Wiener-Hopf equation

$$K(\nu)\, u^-(\nu) = \sigma^+(\nu) + \sigma^-(\nu), \qquad K(\nu) = 4Gm_1 R(\nu)\, J_1^{-2}(\nu) \quad (1.14)$$

Let us represent the function $K(\nu)$ as

$$K(\nu) = K(0)\, K^-(\nu)\, [K^+(\nu)]^{-1} \quad (1.15)$$

$$K(0) = \frac{2G(1+\sigma)}{(1-\sigma)}, \quad K^+(\nu) = \prod_{n=1}^{\infty} \frac{(1 + \nu / a_n)^2}{(1 + \nu / b_n)(1 + \nu / \bar{b}_n)}$$

$$K^-(\nu) = \frac{1}{K^+(-\nu)} \quad (1.16)$$

Here b_n and $\bar{b}_n$ are a pair of complex conjugate zeros of the function $R(\nu)$ ($\operatorname{Re} b_n > 0$, $\operatorname{Im} b_n > 0$). The foundation for the factorization of (1.16) is made in [2], wherein is also obtained the estimate

$$K^-(\nu) = \sqrt{-\nu\, [2 + 2\sigma]^{-1}} + O(1) \quad \text{for} \ \nu \to \infty, \ \operatorname{Re} \nu < 0 \quad (1.17)$$

Taking account of (1.15), let us rewrite (1.14) as

$$K(0)\, K^-(\nu)\, u^-(\nu) = \sigma^+(\nu)\, K^+(\nu) + \sigma^-(\nu)\, K^+(\nu) \quad (1.18)$$

and let us subtract the function

$$N_k(\nu) = K^+(a_k)\, [\sigma^-(\nu) + 2A_k G m_1 a_k^2 (a_k - \nu)^{-1} J_0(a_k)\, T(a_k)] \quad (1.19)$$

$$T(a_k) = \sum_{p=1}^{\infty} [(b_p + a_k)^{-1} + (\bar{b}_p + a_k)^{-1} - 2(a_p + a_k)^{-1}]$$

from both sides of (1.18).

The left and right sides of the equations obtained are regular in corresponding half-planes with the common strip $0 < \operatorname{Re} \nu < \alpha$. On the basis of (1.16), (1.19), conditions (1.3) and the estimate (1.17), it is easy to prove the identities

$$K(0)\, K^-(\nu) u^-(\nu) - N_k(\nu) = \sigma^+(\nu) K^+(\nu) + \sigma^-(\nu) K^+(\nu) - N_k(\nu) = 0 \quad (1.20)$$

hold by customary methods [3] relying on the Liouville theorem.

Hence, and from (1.11) we obtain

$$C_k(\nu) = -\frac{N_k(\nu)}{2G\nu R(\nu)\, K^+(\nu)} = -\frac{N_k(\nu)}{m_4 K(0)\, \nu J_1^2(\nu)} \quad (1.21)$$

The solution of the problem (1.6), (1.7) is given by the inversion formulas

$$u^{k2}(z, r) = \frac{1}{2\pi i} \int_L C_k(\nu)\,[\varepsilon'(\nu) + \rho'(\nu)]\, e^{\nu z} d\nu$$
$$w^{k2}(z, r) = \frac{1}{2\pi i} \int_L C_k(\nu)\,\nu\,[\varepsilon(\nu) - \rho(\nu)]\, e^{\nu z} d\nu \qquad (1.22)$$

where the contour L lies within the strip $0 < \operatorname{Re} \nu < \alpha$.

Combining (1.5) and (1.22), we obtain the solution of the original problem

$$u^{(k)}(z, r) = m_1 a_k e^{a_k z} (A_k z + B_k) J_1(a_k r) + \frac{1}{2\pi i} \int_L C_k(\nu)\,[\varepsilon'(\nu) + \rho'(\nu)]\, e^{\nu z} d\nu$$

$$w^{(k)}(z, r) = - m_1 e^{a_k z} [A_k (a_k z + 4\sigma - 3) + B_k a_k]\, J_0(a_k r) + \frac{1}{2\pi i} \int_L C_k(\nu)\,\nu\,[\varepsilon(\nu) - \rho(\nu)]\, e^{\nu z} d\nu \qquad (1.23)$$

$$\sigma_z^{(k)}(z, r) = 2Gm_1 a_k e^{a_k z} [A_k (m_4 - a_k z) + B_k m_2^{-1} (\sigma a_k + \sigma - 1)]\, J_0(a_k r) + \frac{G}{\pi i} \int_L C_k(\nu)\,\nu^2\,[\varepsilon(\nu) - 4m_1 \rho(\nu)]\, e^{\nu z} d\nu$$

$$\tau_{rz}^{(k)}(z, r) = 2Gm_1 a_k e^{a_k z} [A_k (a_k z - m_2) + B_k a_k]\, J_1(a_k r) + \frac{G}{\pi i} \int_L C_k(\nu)\,\nu\varepsilon'(\nu)\, e^{\nu z} d\nu$$

$$\sigma_r^{(k)}(z, r) = 2Gm_1 a_k e^{a_k z} \{A_k [2\sigma J_0(a_k r) + [a_k J_0(a_k r) - r^{-1} J_1(a_k r)]\, z] + B_k [m_2^{-1} (a_k - \sigma a_k - \sigma) J_0(a_k r) - r^{-1} J_1(a_k r)]\} + \frac{G}{\pi i} \int_L C_k(\nu)\,[r^{-1}\varepsilon'(\nu) + r^{-1}\rho'(\nu) + \nu^2 \varepsilon(\nu)]\, e^{\nu z} d\nu$$

Let us determine the character of the normal stress concentration around the line of separation of the boundary conditions. According to (1.20), we have

$$\sigma_r^{(k)}(z, 1) = \sigma_r^{k1}(z, 1) + \frac{1}{2\pi i} \int_L [\sigma^+(\nu) + \sigma^-(\nu)]\, e^{\nu z} d\nu =$$
$$= \sigma_r^{k1}(z, 1) - \operatorname{Res} \left\{ \frac{N_k(a_k)\, e^{a_k z}}{K^+(a_k)} \right\} + \frac{1}{2\pi i} \int_{L_k} \frac{N_k(\nu)\, e^{\nu z} d\nu}{K^+(\nu)}$$

where L_k denotes the contour L transferred to the right of the point $\nu = a_k$. Let us examine this last expression. Let us close the contour L_k on the right, where the integrand is regular, by a semicircle of large radius. According to the Jordan lemma and the Cauchy theorem, the integral is zero for $z < 0$, and therefore, by virtue of condition (1.2), the first two members cancel. Let us make the substitution $v = \nu z$, without extending it on L_k. According to (1.16), (1.17) and (1.19), we obtain for $z \to +0$

$$\sigma_r^{(k)}(z, 1) = \frac{1}{2\pi i} \int_{L_k} \frac{N_k(\nu)\, e^{\nu z} d\nu}{K^+(\nu)} = \frac{1}{2\pi i} \int \frac{N_k(v/z)\, e^{v} dv}{zK^+(v/z)} =$$
$$= \frac{Gm_1 E_k}{\pi i \sqrt{2(1+\sigma) z}} \int \frac{e^{v} dv}{\sqrt{v}} + O(1)$$

where $E_k = a_k J_0(a_k) K^+(a_k)\{A_k[2\sigma - a_k T(a_k)] + B_k m_2^{-1}(a_k - a_k - \sigma)\}$

Let us close the contour L_k on the left by a semicircle of large radius, and the contour L_* comprised of a two-lipped slit along the negative real semi-axis and the circle $|v| = {}^1/_2 a_1$. Utilizing the Jordan lemma and the Cauchy theorem, let us replace L_k by the contour L_* and by means of the formula

$$\frac{1}{\Gamma(y)} = \frac{1}{2\pi i}\int_{L_*} e^v v^{-y}\,dv \tag{1.24}$$

we finally obtain as $z \to +0$

$$\sigma_r^{(k)}(z, 1) \sim 2Gm_1 E_k [2\pi(1+\sigma) z]^{-1/2} \tag{1.25}$$

Let us determine the shape of the free surface of a deformed cylinder at the line $z = 0$, $r = 1$. According to condition (1.1) and the first part of the relationship (1.20), we have

$$u^{(k)}(z, 1) = \frac{1}{2\pi i}\int_L u^-(\nu) e^{\nu z}\,d\nu = \frac{1}{2\pi i}\int_L \frac{N_k(\nu) e^{\nu z}\,d\nu}{K(0) K^-(\nu)}$$

Let us set $v = -\nu z$ $(z < 0)$ let us take the line $\operatorname{Re} v = -1$ as the contour of integration L_1 in the v plane. Substituting its asymptotics (1.17) in place of the function $K^-(-v/z)$ for small z, and (1.19) in place of $N_k(\nu)$, we obtain

$$u^{(k)}(z, 1) = -\frac{1}{2\pi i}\int_{L_1} \frac{N_k(-v/z) e^{-v}\,dv}{zK(0) K^-(-v/z)} = \frac{E_k \sqrt{-2(1+\sigma) z}}{8\pi(1+\sigma)}\int_{L_1} \frac{e^{-v}\,dv}{v^{3/2}} + O(z^{-1})$$

As before, let us replace L_1 by a contour L_2 comprised of a two-lipped slit along the positive semi-axis and the circle $|v| = {}^1/_2$. Utilizing the formula

$$(e^{2\pi i y} - 1)\Gamma(y) = \int_{L_2} e^{-v} v^{y-1}\,dv \tag{1.26}$$

(the contours in integrals (1.24) and (1.26) are traversed counter-clockwise), we obtain as $z \to -0$

$$u^{(k)}(z, 1) \sim -E_k \sqrt{-z[2\pi(1+\sigma)]^{-1}} \tag{1.27}$$

Now, let us construct a subsystem with a singularity at the point $z = -\infty$. Taking the solution in the form (1.4), (1.8), and satisfying condition (1.2), we obtain [4]

$$u^{k3}(z, r) = \Phi^{k3}\{\varepsilon'(\nu) + \rho'(\nu)\}, \quad w^{k3}(z, r) = \Phi^{k3}\{\nu[\varepsilon(\nu) - \rho(\nu)]\} \tag{1.28}$$

$$\Phi^{k3}\{f(\nu)\} = A_k \operatorname{Re}[f(b_k) e^{b_k z}] + B_k \operatorname{Im}[f(b_k) e^{b_k z}]$$

where $k = -1, -2, \ldots$; $b_{-1}, b_{-2}, \ldots$ are zeros of the function $R(\nu)$ in the $\operatorname{Re}\nu < 0$ half-plane. Let us note that the equalities

$$b_{-k} = -b_k, \quad \varepsilon(-\nu) = -\varepsilon(\nu), \quad \rho(-\nu) = -\rho(\nu) \tag{1.29}$$

hold by definition and by virtue of (1.13), (1.10).

Let us add the solution of the mixed problem

$$\tau_{rz} = \sigma_r = 0 \quad \text{for } r = 1,\ z < 0$$
$$\tau_{rz} = 0, \quad u = -u^{k3}(z, 1) \quad \text{for } r = 1,\ z \geqslant 0$$

to (1.28).

By analogy with the solution (1.22), this solution is found by the Wiener-Hopf method and has the form

$$u^{k4}(z, r) = \Phi^{k4}\{\varepsilon'(\nu) + \rho'(\nu)\}, \quad w^{k4}(z, r) = \Phi^{k4}\{\nu[\varepsilon(\nu) - \rho(\nu)]\}$$

$$\Phi^{k4}\{f(\nu)\} = A_k \operatorname{Re} H_k[f(\nu)] + B_k \operatorname{Im} H_k[f(\nu)] \qquad (1.30)$$

$$H_k[f(\nu)] = -\frac{1}{2\pi i}\int_L \frac{(1+\sigma)\, b_k J_1^2(b_k)\, K^-(b_k)\, f(\nu)\, e^{\nu z}\, d\nu}{\nu R(\nu)\, K^+(\nu)\,(\nu - b_k)}$$

We hence obtain ($k = -1, -2, \ldots$)

$$u^{(k)}(z, r) = \Phi^{(k)}\{\varepsilon'(\nu) + \rho'(\nu)\}, \qquad w^{(k)}(z, r) = \Phi^{(k)}\{\nu[\varepsilon(\nu) - \rho(\nu)]\}$$

$$\sigma_z^{(k)}(z, r) = \Phi^{(k)}\{2G\nu^2[\varepsilon(\nu) - 4m_1\rho(\nu)]\}, \quad \tau_{rz}^{(k)}(z, r) = \Phi^{(k)}\{2G\nu\varepsilon'(\nu)\}$$

$$\sigma_r^{(k)}(z, r) = \Phi^{(k)}\{2G[r^{-1}\varepsilon'(\nu) + r^{-1}\rho'(\nu) + \nu^2\varepsilon(\nu)]\} \qquad (1.31)$$

$$\Phi^{(k)}\{f(\nu)\} = \Phi^{k3}\{f(\nu)\} + \Phi^{k4}\{f(\nu)\}$$

Repeating the discussion applied in deriving formulas (1.27) and (1.25), we correspondingly find the stress concentration and shape of the free surface at the line of separation of the conditions

$$\sigma_r^{(k)}(z, 1) \sim 2G[2(1+\sigma)]^{1/2}(\pi z)^{-1/2}\Phi^{k3}\{K^-(\nu)J_1^2(\nu)\} \quad \text{for} \quad z \to +0 \qquad (1.32)$$

$$u^{(k)}(z, 1) \sim -4(1-\sigma)\sqrt{-2\pi^{-1}(1+\sigma)z}\,\Phi^{k3}\{K^-(\nu)J_1^2(\nu)\} \quad \text{for} \quad z \to -0$$

The principal vector of each homogeneous solution of the subsystems (1.23) and (1.31) is zero. Let us write the elementary solution of the problem of stretching a cylinder [1]

$$\sigma_z = A_0, \quad \tau_{rz} = \sigma_r = \sigma_\varphi = 0, \quad u = -\frac{A_0\sigma r}{2G(1+\sigma)}, \quad w = \frac{A_0 z}{2G(1+\sigma)} + B_0 \quad (1.33)$$

Adding the solution of the mixed problem thereto

$$\tau_{rz} = \sigma_r = 0 \quad \text{for} \quad r = 1,\ z < 0; \quad \tau_{rz} = 0, \quad u = \frac{A_0\sigma r}{2G(1+\sigma)} \quad \text{for} \quad r = 1,\ z \geqslant 0 \quad (1.34)$$

we obtain the solution of the problem (1.1), (1.2) with nonzero principal vector without singularities at the points $z = \pm\infty$

$$u^{(0)}(z, r) = -\frac{A_0\sigma}{2G(1+\sigma)}\left\{1 + \frac{1}{2\pi i}\int_L \frac{[\varepsilon'(\nu) + \rho'(\nu)]\, e^{\nu z}\, d\nu}{2(1-\sigma)\nu^2 J_1^2(\nu) K^-(\nu)}\right.$$

$$w^{(0)}(z, r) = \frac{A_0}{2G(1+\sigma)}\left\{z - \frac{\sigma}{2\pi i}\int_L \frac{[\varepsilon(\nu) - \rho(\nu)]\, e^{\nu z}\, d\nu}{m_4\nu J_1^2(\nu) K^-(\nu)}\right\} + B_0$$

$$\sigma_r^{(0)}(z, r) = \frac{A_0}{4\pi i(1-\sigma^2)}\int_L \frac{[r^{-1}\varepsilon'(\nu) + r^{-1}\rho'(\nu) + \nu^2\varepsilon(\nu)]\, e^{\nu z}\, d\nu}{\nu^2 J_1^2(\nu) K^-(\nu)} \qquad (1.35)$$

$$\sigma_z^{(0)}(z, r) = A_0\left\{1 - \frac{\sigma}{4\pi i(1-\sigma^2)}\int_L [\varepsilon(\nu) - 4m_1\rho(\nu)]\frac{e^{\nu z}\, d\nu}{J_1^2(\nu) K^-(\nu)}\right.$$

$$\tau_{rz}^{(0)}(z, r) = -\frac{A_0\sigma}{4\pi i(1-\sigma^2)}\int_L \frac{\varepsilon'(\nu)\, e^{\nu z}\, d\nu}{\nu J_1^2(\nu) K^-(\nu)}$$

$$\sigma_r^{(0)}(z, 1) \sim \frac{A_0\sigma(1+\sigma)^{1/2}}{(1-\sigma)\sqrt{2\pi z}} \quad \text{for} \quad z \to +0, \quad u^{(0)}(z, 1) \sim -\frac{A_0\sigma\sqrt{-2z}}{G\sqrt{\pi(1+\sigma)}} \quad \text{for} \quad z \to -0$$

In order to construct the system of homogeneous solutions it would be most natural to apply the Wiener-Hopf method directly to problem (1.1),(1.2) and to require that its solution have exponential singularities corresponding to the distribution of zeros of a_k and b_k at the points $z = \pm\infty$. However, such a system would possess a substantial disadvantage. In satisfying the conditions on the endfaces it would generate not normal but quasi- or completely regular systems of algebraic equations depending on the distance between the endfaces and the $z = 0$ plane.

The system (1.23), (1.31), (1.35) reduces to normal Poincaré-Koch systems for an arbitrary combination of the boundary conditions on the flat endface $z = l$, in the domain of contact $z > 0$. On the other endface, the conditions can be posed for $l < 0$ only in combinations of τ_{rz}, w or σ_z, u.

If the method of collocation is utilized, the system of homogeneous solutions can be used for any symmetric conditions on the endfaces, where the endfaces can be not only plane, but also in the shape of a surface of revolution.

2. Let a section of the lateral surface of an infinite cylinder $r = 1$ be compressed by an absolutely rigid cylindrical collar. The collar radius is $1 - \delta$, its length $2l$, there is no friction on the contact surface, and the stresses in the cylinder tend to zero as $z \to \pm\infty$. Taking account of the symmetry of the state of stress, let us write the conditions on the boundary of the appropriate semi-infinite cylinder

$$\tau_{rz} = \sigma_r = 0 \qquad \text{for } r = 1,\ z < 0 \tag{2.1}$$

$$\tau_{rz} = 0, \qquad u = -\delta \qquad \text{for } r = 1,\ 0 \leqslant z \leqslant l \tag{2.2}$$

$$\tau_{rz} = w = 0 \qquad \text{for } 0 \leqslant r \leqslant 1,\ z = l \tag{2.3}$$

Let us seek the solution of the problem (2.1)–(2.3) in the form of the sum

$$u = u^*(z, r) + \sum_{k=0}^{\infty} u^{(k)}(z, r), \qquad w = w^*(z, r) + \sum_{k=0}^{\infty} w^{(k)}(z, r) \tag{2.4}$$

Here $u^*(z, r)$, $w^*(z, r)$ is the solution of the inhomogeneous problem (2.1), (2.2). This latter is a particular case of problem (1.34). Setting $A_0 = -2\delta G(1+\sigma)\sigma^{-1}$ and taking account of (1.33), we obtain from (1.35)

$$u^*(z, r) = \frac{\delta}{2\pi i}\int_L \frac{[\varepsilon'(\nu) + \rho'(\nu)]\, e^{\nu z}\, d\nu}{m_4 \nu^2 J_1^2(\nu) K^-(\nu)}, \qquad w^*(z, r) = \frac{\delta}{2\pi i}\int_L \frac{[\varepsilon(\nu) - \rho(\nu)]\, e^{\nu z}\, d\nu}{m_4 \nu J_1^2(\nu) K^-(\nu)}$$

$$\sigma_r^*(z, 1) \sim -\frac{2\delta G}{1-\sigma}\left(\frac{1+\sigma}{2\pi z}\right)^{1/2} \quad \text{for } z \to +0 \tag{2.5}$$

$$u^*(z, 1) \sim -\delta + \delta\sqrt{-2z\pi^{-1}(1+\sigma)} \qquad \text{for } z \to -0 \tag{2.6}$$

Let us note that (2.6) written in [2] has errors. Let us find the coefficients A_k and B_k. Evidently $A_0 = 0$. Let us close the contour in (2.5) and (1.23) by semicircles passing between negative zeros of the function $J_1(\nu)$. According to the Jordan lemma and the theorem of residues, we obtain

$$w^*(l, r) = \sum_{k=1}^{\infty} \{a_k^{-1} + 2m_1[l + T(a_k)]\}\, \delta J_0^{-1}(a_k) K^+(a_k)\, e^{-a_k l} J_0(a_k r) + 8\delta\sigma m_1 [l + T(0)]$$

$$\tau_{rz}^*(l, r) = \sum_{k=1}^{\infty} 4G\delta m_1 a_k J_0^{-1}(a_k) K^+(a_k)\, [l + T(a_k)]\, e^{-a_k l} J_0(a_k r) \tag{2.7}$$

$$w^{(k)}(l, r) = -m_1 [A_k (a_k l + 4\sigma - 3) + B_k a_k] e^{a_k l} J_0(a_k r) +$$

$$+ \sum_{n=0}^{\infty} d_{kn} [(s_{kn} + p_{kn}) p_{nk} A_k - g_k s_{kn}] e^{-a_n l} J_0(a_n r)$$

$$\tau_{rz}^{(k)}(l, r) = 2Gm_1 a_k [(a_k l - m_2) A_k + B_k a_k] e^{a_k l} J_1(a_k r) +$$

$$+ \sum_{n=1}^{\infty} 2G d_{kn} a_n \{A_k p_{nk} (s_{kn} + p_{kn} - m_4) + g_k (m_4 - s_{kn})\} e^{-a_n l} J_1(a_n r)$$

where

$$s_{kn} = p_{kn} + a_n [l + T(a_n)] + m_2, \quad p_{kn} = a_n (a_k + a_n)^{-1}, \quad a_0 = 0$$

$$d_{kn} = p_{nk} K^+(a_n) K^+(a_k) J_0(a_k) [8(1-\sigma^2) J_0(a_n)]^{-1} \tag{2.8}$$

$$g_k = A_k [2\sigma - a_k T(a_k)] + B_k m_2^{-1} (a_k - a_k \sigma - \sigma)$$

Let us substitute (2.4), and then (2.7) into condition (2.3) and let us interchange the order of summation. Equating coefficients in the functions $J_0(a_k r)$, $J_1(a_k r)$ and introducing the unknowns $X_{k1} = A_k e^{a_k l}$, $X_{k2} = B_k e^{a_k l}$ $(k = 0, 1, \ldots)$, we obtain the infinite system of algebraic equations

$$X_{k1} + \sum_{n=1}^{\infty} 4e^{-l(a_k+a_n)} d_{nk} \{[(s_{nk} + p_{nk} - 1 + \sigma) p_{kn} + (1 - \sigma - s_{nk})(T(a_n) +$$

$$+ 2\sigma) a_n] X_{n1} + (1 - \sigma - s_{nk}) [a_n - a_n\sigma - \sigma] m_2^{-1} X_{n2}\} =$$

$$= -2\delta e^{-a_k l} J_0^{-1}(a_k) K^+(a_k) \{[T(a_k) + l] 4m_1 + a_k^{-1}\} \tag{2.9}$$

$$X_{k2} + \sum_{n=1}^{\infty} 4e^{-l(a_k+a_n)} d_{nk} \{[[(a_k l + 3\sigma - 2)(s_{nk} + p_{nk}) - (1-\sigma)(a_k l + 4\sigma - 3)] p_{kn} +$$

$$+ [(1-\sigma)(a_k l + 4\sigma - 3) - (a_k l + 3\sigma - 2) s_{nk}] [T(a_n) + 2\sigma] a_n] X_{n1} +$$

$$+ [(1-\sigma)(a_k l + 4\sigma - 3) - (a_k l + 3\sigma - 2) s_{nk}] [(1-\sigma) a_n - \sigma] m_2^{-1} X_{n2}\} =$$

$$= -2\delta e^{-a_k l} K^+(a_k) J_0^{-1}(a_k) \{(a_k l - m_2) a_k^{-1} + 4m_1 [T(a_k) + l](a_k l + 3\sigma - 2)\}$$

where $k = 1, 2, \ldots$ and the formula for B_0 is

$$B_0 = -8\delta\sigma [l + T(0)] - 8(1-\sigma^2)^{-1} \sum_{k=1}^{\infty} m_2 J_0(a_k) K^+(a_k)(A_k - g_k) \tag{2.10}$$

The asymptotic expression [2]

$$a_k = k\pi + O(k^{-1}), \qquad b_k = k\pi + iO(\ln k) + O(k^{-1} \ln k) \tag{2.11}$$

and (1.17),(1.19),(2.8) yield such estimates for large k and n

$$p_{kn} < 1, \ K^+(a_k) = O(k^{-1/2}), \ T(a_k) = O(1), \ s_{nk} = O(k), \ d_{nk} = O(k^{-1})$$

Utilizing them we obtain the absolutely convergent series (2.12)

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} M_1 kn \exp[-\pi l(k+n)] \tag{2.13}$$

(M_s are constants) which majorizes the double series of the matrix of the system (2.9).

From the same estimates it follows that the moduli of the free members of the system (2. 9) are bounded. The system (2. 9) is therefore normal [5].

Let us show that its infinite determinant is nonzero. Let us assume $D = 0$. According to the Kronecker-Capelli Theorem 76.7B in [5] there exists a nontrivial solution of the homogeneous system (2. 9) generated by the condition $\delta = 0$, and this contradicts the uniqueness theorem for the solution, bounded at infinity, of the problem of elasticity theory.

Since $D \neq 0$, then according to Theorem 76.5 in [5], a normal solution of the system (2. 9) exists, is unique, and can be obtained by Cramer's rule.

Let us estimate the rapidity of diminution of the coefficients A_n, B_n and the convergence of the series (2. 4). Let us write the system (2. 9) as

$$\sum_{n=1}^{\infty}\sum_{s=1}^{2} a_{kn}{}^{sp} X_{ns} = h_{kp}, \quad p = 1, 2; \qquad k = 1, 2, \ldots \tag{2.14}$$

The elements a_{kn}^{sp} and h_{kp} are easily determined from (2. 9), where if $k \neq n$ and $s \neq p$ simultaneously

$$|a_{kn}{}^{sp}| < M_2 kne^{-\pi l(n+k)}, \quad |h_{kp}| < M_3 ke^{-\pi lk} \tag{2.15}$$

According to Cramer's rule $X_{ns} = D_{ns} D^{-1}$, where D_{ns} is the determinant obtained upon replacing elements of the ns-th column a_{kn}^{sp} by the free terms h_{kp}. Since series in the elements h_{kp} converge absolutely, then D_{ns} is a normal determinant, and by virtue of Theorem 74.9a of [5], an expansion in elements of any row exists for it. Let A_{nq}^{rs} be the cofactor of an element a_{nq}^{rs} if the ns-th row in the determinant D_{ns}, P the Koch majorant for D, and $M_4 = \max |h_{kp}|$ in k and p. The determinant $A_n{}^{rs}$ is not normal, but by relying on Theorems 73.8B, 73.9a and 71.6B in [5], it can be shown that it is bounded: $|A_{nq}^{rs}| < 6M_4P$. Let us expand D_{ns} in elements of the ns-th row. Taking into account that $a_{nn}^{ss} = h_{ns}$ and utilizing the inequalities (2. 15), we obtain

$$|D_{ns}| = \left|\sum_{q=1}^{\infty}\sum_{r=1}^{2} a_{nq}{}^{rs} A_{nq}{}^{rs}\right| < 6M_4P\left|\sum_{q=1}^{\infty}\sum_{r=1}^{2} a_{nq}{}^{rs}\right| < M_5 ne^{-\pi ln}$$

Thus $|X_{ns}| < M_6\, ne^{-\pi ln}$, therefore the series (2. 4) converge no more slowly than a series with general term $k^2 e^{-\pi k(2l-z)}$.

The problem (2. 1) – (2. 3) can also be solved for arbitrarily prescribed functions τ_{rz} and w on the endface. The appropriate example is examined in the next section.

3. Let us consider a problem for a semi-infinite cylinder $r = 1$, $-\infty < z \leqslant l$, which is loaded on the endface and compressed on a section of the lateral surface $0 \leqslant z \leqslant l$ by a cylindrical collar of radius $1 - \delta$. Let us write the boundary conditions

$$\tau_{rz} = \sigma_r = 0 \qquad \text{for } r = 1,\ z < 0 \tag{3.1}$$

$$\tau_{rz} = 0,\ u = -\delta \qquad \text{for } r = 1,\ 0 \leqslant z \leqslant l \tag{3.2}$$

$$\tau_{rz} = f_1(r),\ \sigma_z = f_2(r) \qquad \text{for } 0 \leqslant r \leqslant 1,\ z = l \tag{3.3}$$

Let us seek the solution in the form (2. 4) – (2. 6). Let $f_1(r), f_2(r) \in L_2(0,1)$. Then the following expansions are valid:

$$f_1(r) = \sum_{k=1}^{\infty} c_k J_1(a_k r), \qquad c_k = 2J_0^{-2}(a_k)\int_0^1 f_1(r) J_1(a_k r)\, rdr \tag{3.4}$$

$$f_2(r) = \sum_{k=0}^{\infty} d_k J_0(a_k r), \qquad d_k = 2J_0^{-2}(a_k) \int_0^1 f_2(r) J_0(a_k r)\, r dr$$

where the coefficients c_k and d_k are bounded. Let us find the coefficients A_k and B_k. From the equilibrium condition $A_0 = d_0$ and B_0 remains arbitrary. As in the preceding section, let us compute the inhomogeneous and homogeneous normal stresses at $z = l$

$$\sigma_z^*(l, r) = -\sum_{k=1}^{\infty} 4Gm_1\delta \frac{K^+(a_k)}{J_0(a_k)} \{1 + a_k[l + T(a_k)]\} e^{-a_k l} J_0(a_k r) \tag{3.5}$$

$$\sigma_z^{(k)}(l, r) = 2Gm_1 a_k [A_k(m_4 - a_k l) + B_k m_2^{-1}(\sigma a_k + \sigma - 1) e^{a_k l} J_0(a_k r) +$$

$$+ 2G \sum_{n=1}^{\infty} e^{-a_n l} a_n d_{kn} \{g_k(s_{kn} - m_2) - A_n p_{nk}[s_{kn} + p_{kn} - m_2]\} J_0(a_n r) \qquad (k = 1, 2, \ldots)$$

$$\sigma_z^{(0)}(l, r) = d_0 \left\{1 + \sum_{n=1}^{\infty} 4Gm_1 \frac{K^+(a_n)}{m_2 J_0(a_n)} \{1 + a_n[l + T(a_n)]\} e^{-a_n l} J_0(a_n r)\right.$$

$$\tau_{rz}^{(0)}(l, r) = -\sum_{n=1}^{\infty} 4d_0\sigma m_1 m_3 a_n \frac{K^+(a_n)}{J_0(a_n)} [l + T(a_n)] e^{-a_n l} J_1(a_n r)$$

Let us substitute stresses in the form (2.4) in the left side of the conditions (3.3), utilize the expansions (2.7) for τ_{rz}^* and $\tau_{rz}^{(k)}$ and the expansions (3.5) for the remaining functions. Let us substitute the series (3.4) into the right side of (3.3). Interchanging the order of summation and equating coefficients for the functions $J_1(a_k r)$ and $J_0(a_k r)$, we obtain an infinite system of algebraic equations

$$(a_k l - m_2) X_{k1} + a_k X_{k2} + \sum_{n=1}^{\infty} m_1^{-1} e^{-l(a_k + a_n)} d_{nk} [(s_{nk} + p_{nk} - m_4) p_{kn} X_{n1} +$$

$$+ (m_4 - s_{nk}) g_n^*] = G^{-1} \{2(d_0 m_3 \sigma - \delta G)[l + T(a_k)] J_0^{-1}(a_k) K^+(a_k) e^{-a_k l} + c_k m_4 a_k^{-1}\}$$

$$(m_4 - a_k l) X_{k1} + m_2^{-1}(\sigma a_k + \sigma - 1) X_{k2} + \sum_{n=1}^{\infty} 2m_4 e^{-l(a_k + a_n)} d_{nk} [g_n^*(s_{nk} - m_2) -$$

$$- p_{kn}(s_{nk} + p_{nk} - m_2) X_{n1}] = \frac{m_4}{a_k G} \left\{4m_1(G\delta - d_0 m_3 \sigma) \times\right.$$

$$\left. \times [1 + a_k(T(a_k) + l)] \frac{K^+(a_k)}{J_0(a_k) e^{a_k l}} - d_0 + d_k \right\}$$

where

$$g_n^* = X_{n1}[2\sigma - a_n T(a_n)] + X_{n2} m_2^{-1}(a_n - \sigma a_n - \sigma)$$

In the canonical form (2.9) this system evidently has the matrix (2.15) and bounded free terms. Hence, it and its solution are normal, and therefore, the coefficients A_k and B_k diminish as $e^{-\pi k l}$. The reasoning of the preceding section can be duplicated for $f_1(r) = f_2(r) = 0$, and a stronger estimate can be obtained.

4. Let us consider the contact problem for a semi-infinite cylinder with free lateral surface at the endface under the following boundary conditions:

$$\tau_{rz} = \sigma_r = 0 \qquad \text{for } r = 1,\ l \leqslant z < 0 \tag{4.1}$$

$$\tau_{rz} = 0, \quad u = -\delta \qquad \text{for } r = 1,\ 0 \leqslant z < \infty \tag{4.2}$$

$$\tau_{rz} = f_1(r), \quad w = f_2(r) \quad \text{for} \quad 0 \leqslant r \leqslant 1, \quad z = l \tag{4.3}$$

Let us write the solution as

$$u = u^*(z, r) + \sum_{k=-1}^{-\infty} u^{(k)}(z, r), \quad w = w^*(z, r) + \sum_{k=-1}^{-\infty} w^{(k)}(z, r) \tag{4.4}$$

Utilizing the relationship of generalized Schiff orthogonality [4]

$$\int_0^1 [\varepsilon'(b_k)\rho'(b_n) + \varepsilon'(b_n)\rho'(b_k)]\, rdr = 0 \quad \text{for} \quad k \neq n$$

$$\int_0^1 [\varepsilon'(b_k)\rho'(\bar{b}_n) + \varepsilon'(\bar{b}_n)\rho'(b_k)]\, rdr = 0 \quad \text{for all } k \text{ and } n$$

we expand the functions $f_1(r)$ and $f_2(r)$ in series of homogeneous solutions of the first fundamental problem of elasticity theory for an infinite cylinder

$$f_1(r) = 2G \sum_{k=1}^{\infty} [c_k \varepsilon'(b_k) + \bar{c}_k \varepsilon'(\bar{b}_k)]$$

$$f_2(r) = \sum_{k=1}^{\infty} \{c_k [\varepsilon(b_k) - \rho(b_k)] + \bar{c}_k [\varepsilon(\bar{b}_k) - \rho(\bar{b}_k)]\} \tag{4.5}$$

$$c_k = \int_0^1 \{[f_1(r) - 2Gf_2'(r)]\varepsilon'(b_k) + f_1(r)\rho'(b_k)\}\, rdr \left[4G \int_0^1 \rho'(b_k)\varepsilon'(b_k)\, rdr\right]^{-1}$$

Let us close the contour L in (2.5), (1.30) and (1.31) on the right by semicircles passing between the zeros of the function $R(\nu)$. According to the Jordan lemma and the theorem on residues, we obtain

$$\tau_{rz}^*(l, r) = \sum_{k=1}^{\infty} 2G [t(b_k)\varepsilon'(b_k) + t(\bar{b}_k)\varepsilon'(\bar{b}_k)]$$

$$\tau_{rz}^{(k)}(l, r) = (A_k - iB_k) G \Big\{ b_k \exp(b_k l)\varepsilon'(b_k) + \sum_{n=1}^{\infty} [T(b_k, b_n)\exp(b_n l)\varepsilon'(b_n) + \tag{4.6}$$

$$+ T(b_k, \bar{b}_n)\exp(\bar{b}_n l)\varepsilon'(\bar{b}_n)]\Big\} + (A_k + iB_k) G \Big\{ \bar{b}_k \exp(\bar{b}_k l)\varepsilon'(\bar{b}_k) +$$

$$+ \sum_{n=1}^{\infty} [T(\bar{b}_k, b_n)\exp(b_n l)\varepsilon'(b_n) + T(\bar{b}_k, \bar{b}_n)\exp(\bar{b}_n l)\varepsilon'(\bar{b}_n)]\Big\}$$

$$t(b_k) = \frac{\delta(1+\sigma)\exp(b_k l)}{(1-\sigma) b_k R^*(b_k) K^+(b_k)}, \qquad T(b_k, b_n) = \frac{2(1+\sigma) K^-(b_k) J_1^2(b_k)}{R^*(b_n) K^+(b_n)(b_n - b_k)}$$

$$R^*(b_k) = 2\{b_k J_0^2(b_k) - m_* J_1(b_k)[J_0(b_k) - b_k^{-1} J_1(b_k)]\}$$

Let us substitute the series (4.4) and (4.5), and then (4.6) into the first equation of (4.3). Taking account of (1.29), we interchange the order of summation and equate the coefficients of the functions $\varepsilon'(b_k)$ and $\varepsilon'(\bar{b}_k)$. Introducing the unknowns $Y_{k1} - iY_{k2} = (A_{-k} - iB_{-k})\exp(-b_k l)$ we obtain a normal system of algebraic equations

$$Y_{k1} - \sum_{n=1}^{\infty} \{Y_{n1}\,\mathrm{Re}\,[\varphi_n(b_k) + \varphi_n(\bar{b}_k)] + Y_{n2}\,\mathrm{Im}\,[\varphi_n(b_k) + \varphi_n(\bar{b}_k)]\} = \mathrm{Re}\,\psi_k$$

$$Y_{k2}-\sum_{n=1}^{\infty}\{Y_{n1}\,\mathrm{Im}\,[\varphi_n(\bar b_k)-\varphi_n(b_k)]+Y_{n2}\,\mathrm{Re}\,[\varphi_n(b_k)-\varphi_n(\bar b_k)]\}=-\,\mathrm{Im}\,\psi_k$$

where

$$\varphi_n(b_k)=b_k^{-1}\exp[l(b_k+b_n)]\,T(-b_n,b_k),\quad \psi_k=2b_k^{-1}[t(b_k)-c_k]$$

The second condition in (4.3) is satisfied automatically. In the general case $A_k, B_k \sim O(e^{-\pi kl})$ if $f_1(r)=f_2(r)=0$, which corresponds to compression of an infinite cylinder by two semi-infinite collars $A_k, B_k \sim O(ke^{-2\pi kl})$.

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BUCKLING OF PLATES MADE OF A NEO-HOOKEAN MATERIAL IN THE CASE OF AFFINE INITIAL DEFORMATION

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We construct two-dimensional equations describing the bending bifurcation of equilibrium of plates made of a neo-Hookean material, for the case of homogeneous initial deformation. We derive three-dimensional equations of neutral equilibrium for this material. A variational principle which is equivalent to the differential equations of neutral equilibrium and analogous to the the Reissner's principle in the classical theory of elasticity, is established. We use this variational principle to derive two-dimensional equations of buckling of plates by approximating the variations in the values of the unknown functions in the normal direction. The cases of buckling of a uniformly compressed circular plate and of a rectangular plate under a combined load are used as examples. An exact solution of three-dimensional equations of neutral equilibrium is obtained for a circular cylinder compressed over its lateral surface, with axial symmetry present, and compared with the corresponding two-dimensional result.

1. Equations of neutral equilibrium for a neo-Hookean material. Specific potential energy of deformation is given for a neo-Hookean material by the following expression:

$$W=c_1(I_1-3),\qquad c_1=\mathrm{const}$$